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COEXISTENCE, COMPETITION AND HYPERCYCLIC INTERACTION IN SOME SYSTEMS OF BIOLOGICAL INTEREST

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Systems of differential equations which describe the selective behavior of self-replicating macromolecules or species under the constraint of constant organization are classified according to their coexistent and competitive behavior as well as the cooperative and/or hypercyclic nature of the inter-species interactions. Two theorems are proved which relate the possibilities for coexistence and competition among species to the presence or absence of hypercyclic linkages.

1. Introduction

We wish to classify and briefly analyze some systems of differential equations which have been used to describe processes of selection among biological macromolecules or species. We consider a system of n species $x_1, x_2 \dots x_n$ whose time development is governed by differential equations of the form

$$\dot{x}_i = \Gamma_i(x), \quad i = 1, 2, \dots, n. \tag{1}$$

We assume that one may ignore the effects of any spatial inhomogeneities.

For the systems of interest here, the term Γ_i is proportional to the concentration of x_i; the rate of growth or decay of a population depends upon that population level. We may thus write

$$\dot{x}_i = \theta_i(x) x_i, \quad i = 1, 2, \dots, n. \tag{2}$$

We are also interested in systems in which the total population Σx_i is fixed, e.g., by suitable fluxes into and out of the system. Such systems are said to be subject to a constraint of constant organization. If θ_i takes the form

$$\theta_i(x) = \Lambda_i(x) - \phi(x), \tag{3}$$

where

$$\phi(x) = \frac{1}{c} \sum_{i=1}^{n} x_i \Lambda_i(x) \tag{4}$$

and

$$c = \sum_{i=1}^{n} x_i \tag{5}$$

then substitution of eqs. (2–5) in eq. (1) shows that

$$\sum \dot{x}_i = 0 \tag{6}$$

so that the total population is stationary, as desired, at the value c of eq. (5).

Equations of the above form were introduced by Eigen [1] and have been extensively studied by Eigen and Schuster [2] in conjunction with the problem of self-organization of systems of self replicative units. By choosing $\Gamma_i(x)$ to be of the form of eq. (2) we restrict ourselves to systems without mutation or, more generally, we consider the x_i to represent quasi-species [3,4] rather than individual species.

2. Definitions

We define x_i to have a *cooperative, noncooperative,* or *anticooperative effect* on x_j (at a given x) according to whether $\partial\Lambda_j/\partial x_i$ is greater than, less than or equal

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to zero. A system may be defined to be *noncooperative* if there are no cooperative or anticooperative effects present. Systems which are not noncooperative are defined to be *cooperative, anticooperative* or *mixed* according to whether all non-zero $\mathrm{d}\Lambda_j/\mathrm{d}x_i$ are positive, all are negative, or some are positive and some negative, respectively. Note that an individual effect or a system may change its nature as the vector x changes.

Species x_j which have only noncooperative effects on other species ($\partial\Lambda_i/\partial x_j = 0$ for all $i \neq j$) will be called *end species.* Any species which has a cooperative or anticooperative effect on itself ($\partial\Lambda_j/\partial x_j \neq 0$) will be said to be *hyperbolic.* The hyperbolic interaction may be either *positive* ($\partial\Lambda_j/\partial x_j > 0$) or *negative* ($\partial\Lambda_j/\partial x_j < 0$).

A system of equations will be said to be *degenerate* if there exist $i \neq j$ such that

$$\Lambda_i(x) = \Lambda_j(x)$$

for all $x \in X_n$, where we define the *simplex* X_n [2] as

$$X_n = \left\{ x : \sum_{i=1}^{n} x_i = c,\ 0 \leqslant x_i \leqslant c,\ i = 1, 2, ..., n \right\}. \qquad (7)$$

All x of physical interest must lie in X_n.

We shall be particularly interested in the question of how many species will survive in a given system as $t \to \infty$. In this connection, we define a system to be *competitive* (at a given value of c) if it possesses a stable fixed point at one or more of the vertices of the simplex X_n, i.e., at a point $x_i = c$, $x_j = 0$ for $j \neq i$. In such systems, at least some initial conditions lead ultimately to the extinction of all but one of the species.

A *coexistent system* is defined as one in which the system of eqs. (1–5) has a stable attractor (fixed point or limit cycle) lying in the interior of X_n. Such systems allow for the long-term survival of all species, $x_1, x_2, ..., x_n$ under at least some initial conditions. A *partially coexistent system* is one having a stable attractor which lies on a hyperface, but not on a vertex of S_n (i.e., $x_i = 0$ for at least one but not more than $n - 2$ of the x_i). In these systems, some species will die out, but two or more can survive.

In many cases it is of interest to know how the relative populations of species change when the total population of the system is allowed to increase. We define a *pseudo-coexistent system* as a coexistent system in which one or more of the ratios x_i/x_j at the stable attractor approaches infinity (or zero) as $c \to \infty$. If all x_i/x_j at the attractor remain finite and nonzero as c increases without bound, then the system will be said to be *totally coexistent.* It is these systems which are of greatest interest in discussing the evolution of natural self-organization [2].

We note that a given system may be both competitive and coexistent at a particular value of c, the ultimate fate of the system depending then upon the initial conditions.

A system will be said to be *hypercyclic* if there exists a sequence of integers $i_1, i_2, ..., i_n$ such that $\{i_1, i_2, ..., i_n\} = \{1, 2, ..., n\}$ and

$$\frac{\partial\Lambda_{i_n}}{\partial x_{i_1}} \prod_{j=1}^{n-1} \frac{\partial\Lambda_{i_j}}{\partial x_{i_{j+1}}} \neq 0. \qquad (8)$$

A system is *partially hypercyclic* if it contains a subset $x_{k_1}, x_{k_2} ... x_{k_q}$, $2 \leqslant q \leqslant n-1$, such that $k_r \neq k_s$ if $r \neq s$ and

$$\frac{\partial\Lambda_{k_q}}{\partial x_{k_1}} \prod_{j=1}^{q-1} \frac{\partial\Lambda_{k_j}}{\partial x_{k_{j+1}}} \neq 0. \qquad (9)$$

A system which is neither hypercyclic nor partially hypercyclic will be called *non-hypercyclic.*

3. A theorem on coexistence

Eigen and Schuster [2] have shown that a specific class of systems which they call catalytic chains are pseudocoexistent, while a certain category of hypercyclic systems which they designate hypercycles are totally coexistent. We now wish to establish a somewhat more general result in the following *Theorem:*

Any non-degenerate system which does not contain hyperbolic species must be at least partially hypercyclic if it is to be totally coexistent.

We first prove a lemma which is almost trivial when considered in graphical terms.

Lemma: A non-hypercyclic system must contain at least one species x_i such that

$$\partial\Lambda_j/\partial x_i = 0 \quad \text{for all } j \neq i \qquad (10)$$

(Non-hypercyclic systems must contain end species).

Proof: By induction on n. The lemma holds trivially for $n = 2$. Suppose that it is true for $n = N$, and assume that a nonhypercyclic system exists with $n = N + 1$, and no x_i which obeys eq. (10), i.e., for every x_i, there exists some $j \neq i$ such that

$$\partial\Lambda_j/\partial x_i \neq 0. \tag{11}$$

Consider now the set $R_0 = \{x_i : 1 \leqslant i \leqslant N\}$, which by assumption is an N-membered non-hypercyclic system. It therefore contains by the inductive hypothesis at least one x_q with the property that

$$\partial\Lambda_j/\partial x_q = 0 \quad \text{for all } j \neq q,\ 1 \leqslant j \leqslant N \tag{12}$$

We divide R_0 into two subsets:

$$R_1 = \{x_q \in R_0 : \partial\Lambda_j/\partial x_q = 0 \quad \text{for all } j \neq q,\ 1 \leqslant j \leqslant N\}, \tag{13}$$

$$T_1 = R_0 - R_1. \tag{14}$$

Since our assumption requires that every x_q have a non-zero effect on some Λ_j (eq. 11), eq. (13) implies that

$$\partial\Lambda_{N+1}/\partial x_q \neq 0 \quad \text{for all } x_q \in R_1. \tag{15}$$

Eq. (15) in turn implies that

$$\partial\Lambda_q/\partial x_{N+1} = 0 \quad \text{for all } x_q \in R_1 \tag{16}$$

because otherwise the subset $\{x_q, x_{N+1}\}$ would fulfill the conditions of eq. (9) and be hypercyclic.

If the set T_1 is empty, then the lemma is proven, because x_{N+1} is the x_i of eq. (10). If not, we divide T_1 into two subsets:

$$R_2 = \{x_q \in T_1 : \partial\Lambda_{j_q}/\partial x_q \neq 0 \quad \text{for some } x_{j_q} \in R_1\}, \tag{17}$$

$$T_2 = T_1 - R_2. \tag{18}$$

Again, our assumption requires that

$$\partial\Lambda_q/\partial x_{N+1} = 0 \quad \text{for all } x_q \in R_2 \tag{19}$$

since otherwise $\{x_q, x_{j_q}, x_{N+1}\}$ would form a hypercyclic subsystem.

Again, if T_2 is empty we have proven the lemma. If not we further divide T_2 into R_3 and T_3, or in general, we divide T_m into

$$R_{m+1} = \{x_q \in T_m : \partial\Lambda_q/\partial x_q \neq 0 \quad \text{for some } x_{j_q} \in R_m\}, \tag{20}$$

$$T_{m+1} = T_m - R_{m+1}. \tag{21}$$

The arguments used above imply that

$$\partial\Lambda_q/\partial x_{N+1} = 0 \quad \text{for all } x_q \in R_{m+1}. \tag{22}$$

R_{m+1} cannot be empty so long as T_m is non-empty, since by the assumption each x_q in T_m must interact with some x_j and by construction this x_j cannot be either x_{N+1} or a member of any of the R_i for $i \leqslant m - 1$. Since by the inductive hypothesis there must be at least one x_q in T_m which does not interact with any x_j in T_m, that x_q must interact with a member of R_m, and hence R_{m+1} cannot be empty. Clearly then, since R_0 had only N members, T_{m+1} must be an empty set for some $m \leqslant N$, and the lemma is proven because

$$\partial\Lambda_j/\partial x_{N+1} = 0 \quad \text{for all } x_j \in R_0.$$

Having established the lemma, we now complete the proof of the theorem by showing that in any n-membered system which is not at least partially hypercyclic, there must exist at least one pair of species x_i, x_j such that $x_i/x_j \to \infty$ for any fixed point of the system as $c \to \infty$.

According to eqs. (2–5), any fixed point y of the system is characterized by the equations

$$(\Lambda_i(y) - \phi(y))y_i = 0 \quad i = 1, 2, ..., n, \tag{23}$$

$$\sum_{i=1}^{n} y_i = c, \tag{24}$$

where $\phi(y)$ is given by eq. (4).

Total coexistence requires a fixed point y in the interior of S_n, i.e., a fixed point such that $y_i \neq 0$ for $1 \leqslant i \leqslant n$. For such a point, eqs. (23) and (4) can be replaced by the $n - 1$ independent (so long as the system is non-degenerate) equations

$$\Lambda_i(y) = \Lambda_1(y) \quad i = 2, 3, ..., n. \tag{25}$$

Eqs. (25) plus eq. (24) then constitute n independent equations in the n unknowns $y_1, y_2, ..., y_n$ and their solutions, if any, in the interior of X_n correspond to possible stable coexistent states or possible centers of stable coexistent limit cycles.

If the system under consideration is not at least pseudohypercyclic, then we can according to the lemma, define a non-empty set K:

$$K = \{i : \partial\Lambda_j/\partial x_i = 0 \text{ for all } j \neq i,\ 1 \leqslant j \leqslant n\}, \tag{26}$$

If K includes the entire set $\{1, 2, ..., n\}$ then the system consists of independent competitors and, as shown elsewhere [5], its only stable asymptotic solutions lie at the vertices of the simplex S_n. If K is a proper subset, then define

$$\bar{K} = \{1, 2, ..., n\} - K. \tag{27}$$

By the definition of K, eq. (26), and the absence of hyperbolic species, the Λ_j can depend only on the x_i for which $i \in \bar{K}$. Suppose that $\bar{K}$ has m members, $1 \leqslant m \leqslant n - 1$. Then we can choose m of the equations (25) which involve all the i in $\bar{K}$. These are now m independent equations for the m y_i such that $i \in \bar{K}$, and these equations determine those y_i independent of c. Let $r = \Sigma_{i\in\bar{K}} y_i$. Then by virtue of eq. (24)

$$\sum_{i\in K} y_i = c - r, \tag{28}$$

and at least one $y_i \in K$ must equal or exceed $(c - r)/(n - m)$ for all values of c. Since for $j \in \bar{K}, y_j$ is independent of c, given any M we can always choose a c large enough so that for some $i \in K$ and any $j \in \bar{K}$,

$$y_i/y_j \geqslant (c - r)/r(n - m) > M, \tag{29}$$

which implies that the system cannot be totally coexistent, completing the proof of the theorem.

4. A theorem on competition

Having established a connection between hypercyclic and totally coexistent systems, we now consider one aspect of the relation between competitive and hypercyclic interactions in the following *Theorem:*

Let S be a hypercyclic system with no positive hyperbolic interactions, which contains a sequence $x_{i_1}, x_{i_2}, ..., x_{i_q}$ *as in eq. (8) with the additional property that all the interactions have a positive lower bound, i.e.*

$$\partial\Lambda_{i_n}/\partial x_{i_1} \geqslant g_1 > 0, \tag{30a}$$

$$\partial\Lambda_{i_j}/\partial x_{i_{j+1}} \geqslant g_{j+1} > 0 \quad j = 1, 2, ..., n - 1, \tag{30b}$$

for all x. Then there exists a critical value c_0 *such that for* $c > c_0$*, S is not competitive.*

Proof: Let z_i be the point at which $x_i = c$ and all other $x_j = 0$. From eqs. (2–5) it is clear that all n of the z_i are fixed points of the system for all values of c.

From eqs. (2–5) we obtain the jacobian matrix of the system, which has elements

$$A_{ij} \equiv \partial\dot{x}_i/\partial x_j = \delta_{ij}(\Lambda_i - \phi) + x_i\left(\frac{\partial\Lambda_i}{\partial x_j} - \frac{1}{c}\left(\Lambda_j + \sum_k x_k \frac{\partial\Lambda_k}{\partial x_j}\right)\right), \tag{31}$$

where δ_{ij} is the Kronecker delta.

At the point z_k, A_{ij} takes the form

$$A_{ij} = \delta_{ij}(\Lambda_i(z_k) - \Lambda_k(z_k)) - \delta_{ik}\Lambda_j(z_k). \tag{32}$$

The eigenvalues of this matrix are just

$$\lambda_k^{(i)} = \Lambda_i(z_k) - \Lambda_k(z_k) \quad \text{for } i \neq k, \tag{33a}$$

$$\lambda_k^{(k)} = -\Lambda_k(z_k). \tag{33b}$$

For z_k to be a stable fixed point, all the $\lambda_k^{(j)}$ must have negative real parts. S is competitive if any of the z_k is stable. We have therefore established the following *Lemma:*

The system S is competitive if and only if for at least one k, $1 \leqslant k \leqslant n$,

$$\Lambda_k(z_k) > 0 \tag{34a}$$

and

$$\Lambda_k(z_k) > \Lambda_i(z_k) \quad \text{for } i \neq k. \tag{34b}$$

We now let

$$\Lambda_{i0} \equiv \Lambda_i(\mathbf{0}), \tag{35}$$

where $\mathbf{0}$ is the point with all components zero. Consider the point z_{i_j} where the sequence $\{i_j\}$ is that of eq. (30). Since there are no positive hyperbolic species, it must be true that

$$\Lambda_{i_j}(z_{i_j}) \leqslant \Lambda_{i_j 0}. \tag{36}$$

By virtue of eqs. (30), it must also be true that

$$\Lambda_{i_{j-1}}(z_{i_j}) \geqslant \Lambda_{i_{j-1} 0} + cg_{i_j}, \tag{37}$$

where if $j = 1$ we replace $j - 1$ by n in eq. (37). Therefore, if

$$c > (\Lambda_{i_j0} - \Lambda_{i_{j-1}0})/g_{i_j} \equiv c_j, \tag{38}$$

then the point z_{i_j} is not stable. If we now choose

$$c_0 = \inf_{1 \leqslant j \leqslant n} (c_j),$$

then for any $c > c_0$ all fixed points (z_{i_j}) must be unstable and the system cannot be competitive, thus proving the theorem.

5. Discussion

The coexistence theorem constitutes a generalized version of a result shown by Eigen and Schuster [2] that nonhypercyclic systems with only linear terms in the Λ_i can be at most pseudocoexistent. These authors then argue that total coexistence in physically reasonable cooperative systems requires hypercyclic interaction. The present results place that assertion in a somewhat more general context.

The set K defined in eq. (26) is the set of end species, i.e., those which in the language of Eigen and Schuster lie at the ends of catalytic chains. The lemma shows that systems without hypercyclic subsystems must possess one or more end species. The theorem is then proved by observing that as c increases the asymptotic populations of the non-end species remain constant, while at least one end species grows in population as c grows. Thus as the system increases its size, it must eventually be dominated by one or more end species.

We note that the restriction to non-hyperbolic species may be relaxed to refer only to the end species. Hyperbolic growth terms in the members of $\overline{K}$ will not affect the arguments which led to eq. (29). More specifically, it would appear that only negative hyperbolic interactions ($\partial\Lambda_i/\partial x_i < 0$) for the end species could lead to total coexistence, since positive terms of this form should only serve to increase the dominance of the end species at high c.

For example, a 2-species catalytic chain with quadratic decay of the end species:

$$\mathrm{d}x_1/\mathrm{d}t = (k_1 - \phi)x_1, \tag{39a}$$

$$\mathrm{d}x_2/\mathrm{d}t = (k_2 x_1 - k_3 x_2 - \phi)x_2, \tag{39b}$$

is cooperative, non-competitive and totally coexistent for $c > k_1/k_2$. Epstein [6] has considered other nonhypercyclic systems with negative hyperbolic interactions and has shown several of them to be totally coexistent even when the system is noncooperative. It thus appears that long term stable coexistence of an entire array of self-reproducing species during the growth of the system requires either catalytic linkages which allow for a cyclic feedback of information or a superlinear mechanism for limiting the growth of any species at the end of a catalytic chain.

The hypercycles discussed by Eigen and Schuster [2] obey eqs. (30) when they consist of self-reproducing species ($\Lambda_i = k_i + k_i' x_j$, $k_i > 0$). The conditions of the competition theorem can be relaxed somewhat to require eqs. (30) to hold only outside a given neighborhood of the origin. The result obtained then applies to Eigen and Schuster's "pure" ($k_i = 0$) hypercycles as well. Thus hypercycles, as well as a more general class of hypercyclic systems, are inherently noncompetitive once the system reaches a certain critical size.

The results of Eigen and Schuster [2] suggest that while partially hypercyclic systems may in principle allow for survival of all species according to our theorem, in fact non-hypercyclic side chains should eventually either die out or destroy the hypercyclic subsystem. The approaches employed here are insufficient to prove this conjecture in general, but we feel that it should hold for a rather wide range of systems.

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